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ON THE CONDITIONED COOLING AND THE CUBICAL CONTRACTION OF A HOMOGENEOUS SPHERE.*

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13. In the former paper (ANNALS OF MATHEMATICS, Vol. III. No. 3) we have discussed the problem of the "free cooling" of a homogeneous sphere. In the present paper it is proposed to consider what we have, for brevity, called the case of "conditioned cooling of a homogeneous sphere," or the cooling of such a sphere in a diathermous medium. In addition, we shall give an application of the theory of cooling in determining under certain limitations the consequent cubical contraction in large spheres, this problem being of special interest with reference to the secular contraction of the earth.

The physical circumstances, and the mathematical features of the solution of the problem of conditioned cooling have been very fully and ably set forth by Fourier,† Poisson,‡ and Riemann.§ So far as the purely analytical requirements are concerned, their works, especially that of Poisson, will be found exhaustive. We need, therefore, only to refer to them for a complete exposition of the elementary considerations which lead to their final result, our object being to adapt that result to the requirements of the computer, having in mind always an application of the theory to the earth. It will be desirable, however, for the sake of clearness and definiteness to make a formal statement of the fundamental assumptions and relations involved in the steps of the investigation which precede those we have in view.

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* Since this paper is essentially a continuation of the one on "The Free Cooling of a Sphere," published in Vol. III. No. 3 of the ANNALS OF MATHEMATICS, the section numbers and those designating the principal equations are made continuous with the corresponding numbers of the earlier paper. The notation introduced in the latter has been adhered to as far as possible.

† Théorie Analytique de la Chaleur. Paris, 1822. Analytical Theory of Heat, by Joseph Fourier. Translated, with notes, by Alexander Freeman. Cambridge, 1878.

‡ Théorie Mathématique de la Chaleur. Paris, 1835.

§ Partielle Differentialgleichungen und deren Anwendung auf physikalische Fragen. Vorlesungen von Bernhard Riemann, für den Druck bearbeitet und herausgegeben von Karl Hattendorff. Dritte Auflage. Braunschweig, 1882.

14. The assumptions made are four, viz.: first, a constant coefficient of diffusion for the entire mass of the sphere; second, an initial uniform excess in temperature above that of the surrounding medium of the entire mass of the sphere; third, a rate of flow of heat from the surface of the sphere to the medium proportional to their difference in temperature; and fourth, a constant temperature for the surrounding medium.

As in the previous paper, let r be the radius of any spherical shell whose thickness is dr, u the excess in temperature over that of the surrounding medium of this shell at the time t after the initial epoch, and a^2 the coefficient of diffusion. Then these quantities must satisfy the partial differential equation

$$\frac{\partial (ru)}{\partial t} = a^2 \frac{\partial^2 (ru)}{\partial r^2}, \tag{29}$$

and the solution of the problem consists in determining the integral of this equation, (ru) being regarded as the unknown quantity, subject to certain relations which we proceed to set down. If u_0 be the initial uniform excess of the temperature of the sphere over that of the surrounding medium, we must have

$$ru = ru_0 \text{ for } t = 0. \tag{30}$$

In order that u may not be infinite when $r = 0$, we must have

$$ru = 0 \text{ for } r = 0. \tag{31}$$

It remains to express the characteristic condition of the problem, namely, that which defines the emission of heat at the surface of the sphere. If k be the coefficient of conductivity for the sphere, the quantity of heat which passes across a unit of its surface in an element of time is expressed by the value of $-k\frac{\partial u}{\partial r}$ when $r = r_0$, the radius of the surface of the sphere. On the other hand, this quantity of heat is equal to that which the surrounding medium can carry away, or according to our third and fourth assumptions, to Hu, H* being a constant and u the temperature corresponding to $r = r_0$. Hence we have

$$-k\frac{\partial u}{\partial r} = Hu,$$

or, if we put $H/k = h$,

$$\frac{\partial u}{\partial r} + hu = 0 \text{ for } r = r_0.$$

But since $u = \frac{(ru)}{r}$, $$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial (ru)}{\partial r} - \frac{u}{r},$$

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*This quantity is called by Sir W. Thomson the emissivity. It is the quantity of heat emitted per unit of time, per unit of surface, per unit of excess of temperature; and may be due to conduction, convection, or to both. See §71, Article Heat, Encyclopedia Britannica.

which, substituted in the above relation, gives

$$\frac{\partial (ru)}{\partial r} + (rh - 1)\,u = 0 \text{ for } r = r_0. \tag{32}$$

Now a particular integral of (29) satisfying the relation (31) is

$$ru = C' e^{-a^2 t \lambda^2} \sin \lambda r, \tag{33}$$

in which C' is a constant, and λ a factor such that

$$\frac{\partial (ru)}{\partial r} = C' \lambda e^{-a^2 t \lambda^2} \cos \lambda r$$

and

$$u = C' r^{-1} e^{-a^2 t \lambda^2} \sin \lambda r$$

will satisfy (32). Substituting these values in (32) we have λ defined by the following transcendental equation: —

$$r_0 \lambda \cos r_0 \lambda + (r_0 h - 1) \sin r_0 \lambda = 0. \tag{34}$$

This equation has an infinite number of unequal real roots,* and since any one of them will satisfy (33), the complete integral of (29) must consist of a series of terms of the form (33). That is,

$$ru = \sum_{n=1}^{n=\infty} C'_n e^{-a^2 t \lambda_n^2} \sin \lambda_n r. \tag{35}$$

In this, $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the successive roots of (34). The constants C'_1, C'_2, $C'_3, \ldots$ must be such that (35) will satisfy the initial relation (30); i. e. making $t = 0$ in (35), the relation

$$ru_0 = \sum_{n=1}^{n=\infty} C'_n \sin \lambda_n r.$$

The process of determining C'_n from this equation consists in multiplying both sides by $\sin \lambda_n r dr$ and integrating between the limits 0 and r_0, account being taken of the relation (34). We shall not go through the steps of this process, referring the reader to Poisson and Riemann for the details, but shall pass immediately to one of the usual final forms for (35). This form is†

$$ru = \frac{2u_0}{r_0} \sum_{n=1}^{n=\infty} \frac{1}{\lambda_n^2} \cdot \frac{r_0^2 \lambda_n^2 + (r_0 h - 1)^2}{r_0^2 \lambda_n^2 + r_0 h (r_0 h - 1)} e^{-a^2 t \lambda_n^2} (\sin r_0 \lambda_n - r_0 \lambda_n \cos r_0 \lambda_n) \sin r \lambda_n. \tag{36}$$

15. The expression (36) satisfies all of the relations (29) to (32), because each of its terms satisfies them, and is the complete solution of the problem. But, though it meets all the demands of pure analysis, its form is quite repulsive to

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* See any of the works first cited above.

† See Riemann, Differentialgleichungen, §69. Other equivalent forms are given by Fourier, Poisson, and Riemann.

the computer or to any one who may desire to use it, considering that the quantities λ_n must be determined from the transcendental equation (34). Some attempts have been made to simplify this expression, but the results are not satisfactory, inasmuch as they require either a time so great that we may neglect all terms of the series after the first, or a very small radius of the sphere, or a very great radius of the sphere. No attempt, so far as we know, has been made to transform the series into another more converging or tractable, though Poisson has summed it to a limited degree of approximation, as we shall presently explain. Neither is the improvement we have to suggest entirely satisfactory, for it is restricted to the case where $r_0 h$ of (34) is greater than unity,* but this is the case with which we are most particularly concerned and that to which we now direct attention.

16. The quantity h in (34) is of -1 dimension in length,† and hence $r_0 h$ is of zero dimension or an abstract number.

For brevity, let us put

$$x = 1 / (r_0 h) \quad \text{and} \quad y_n = r_0 \lambda_n.$$

Then equation (34) gives

$$\tan y_n = - \frac{x y_n}{1 - x}. \tag{37}$$

Likewise, if we make the same substitutions in (36), it may be put in the more compact form

$$ru = 2 r_0 u_0 \sum_{n=1}^{n=\infty} \frac{1}{y_n^2} \cdot \frac{(1-x)^2 + x^2 y_n^2}{(1-x) + x^2 y_n^2} \cdot \frac{\sin y_n}{x} \cdot e^{-a^2 t \lambda_n^2} \sin \lambda_n r. \tag{38}$$

Now it will be observed that when $x = 0$, $y_n = n\pi$, $\lambda_n = n\pi / r_0$, and

$$\frac{\sin y_n}{x} = \frac{0}{0} = - n\pi \cos n\pi;$$

hence (38) becomes

$$ru = \frac{2 r_0 u_0}{\pi} \sum_{n=1}^{n=\infty} \frac{(-1)^{n+1}}{n} e^{-a^2 t (n\pi / r_0)^2} \sin n\pi \frac{r}{r_0}.$$

This agrees with equation (10), which expresses the solution in the case of free cooling. The case of conditioned cooling, therefore, approximates to that of free cooling in proportion to the smallness of the number x. This number may be small by reason of the largeness of either r_0 or h, or of both. From certain observations of underground temperatures in the garden of the Paris Observatory, Poisson concluded that for the earth in the vicinity named, h is a little greater than 1, using the metre as space unit.* This would make $x = 1 / (r_0 h)$ about $1 / 6{,}300{,}000$. Whether this value is nearly correct as applied to the whole surface of the earth may be doubted, but x might be 10,000 times as great without materially affecting the applicability to the earth of the process we shall develop. The correctness of that process only requires that $x < 1$.

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* It is obvious, however, that a slight modification of the process followed in the text may be used in treating the case where $r_0 h < 1$.

† See Fourier's Section IX. chap. II. p. 130, Freeman's translation. Our h is the same as Fourier's h / K of the page just cited, but the same as his h defined on p. 268.

It is proposed, now, to expand the second member of (38) by Maclaurin's theorem in a series of ascending powers of x. As a preliminary, though not essential part of this work, we may expand y_n of (37) as a function of x and arrive thus at a formula of use in a future numerical application.

17. When $x = 0$, (37) gives $y_n = n\pi$, and this is the first term in the expansionof y_n. To get the coefficients of the terms involving powers of x, we need $\dfrac{d^i y_n}{dx^i}$ for $x = 0$, i being any positive integer. If we differentiate (37) once, and then eliminate $\sec^2 y_n$, the result is

$$(1 - x + x^2 y_n^2)\frac{dy_n}{dx} + y_n = 0.$$

If now we put $Y = y_n^2 \dfrac{dy_n}{dx}$, and differentiate the above equation successively, we find that for $x = 0$

$$\frac{d^i y_n}{dx^i} = (i - 2)\frac{d^{i-1} y_n}{dx^{i-1}} - (i - 1)(i - 2)\frac{d^{i-3} Y}{dx^{i-3}}. \tag{39}$$

By a duplicate application of the theorem of Leibnitz, we can derive a general expression for the $(i - 3)$th differential coefficient of y with respect to x, but its usefulness hardly justifies the space required by its rather complex character. We shall simply write down the first six values of (39). They are

$$\begin{aligned}
\frac{d^i y_n}{dx^i} &= - \quad n\pi && \text{for } i = 1,\\
&= \quad 0 && = 2,\\
&= + \quad 2\,(n\pi)^3 && = 3,\\
&= - \quad 8\,(n\pi)^3 && = 4,\\
&= - \quad 24\,(n\pi)^5 && = 5,\\
&= + \; 384\,(n\pi)^5 && = 6.
\end{aligned}$$

Hence by Maclaurin's theorem we have

$$y_n = n\pi\left(1 - x + \tfrac{1}{3}n^2\pi^2 x^3 - \tfrac{1}{3}n^2\pi^2 x^4 - \ldots\right). \tag{40}$$

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*Théorie de la Chaleur, p. 502. The value given is $b = h = 1.05719$.

Neglecting terms involving powers of x higher than the 4th, this equation may be written

$$\begin{aligned} y_n &= n\pi(1-x)(1+\tfrac{1}{3}n^2\pi^2x^3) \\ &= n\pi - n\pi x\{1-\tfrac{1}{3}n^2\pi^2x^2(1-x)\}, \end{aligned} \tag{41}$$

one of which forms will be found convenient for logarithmic computation of the smaller values of y_n. For the larger values, the series (40) will, in general, prove too slowly converging, but it is not necessary to calculate such values. Our only use for the smaller values is a numerical verification.

18. To develop (38) we observe that each of the seven factors under the sign of summation, is a function of x. We might, therefore, expand these factors separately and then find their product. On the other hand, we might get the differential coefficients of (ru) with respect to x by direct differentiation. We shall adopt a combination of these two processes.

First, develop the product $x^{-1}\sin y_n$. By means of (40), we readily find

$$x^{-1}\sin y_n = -n\pi\cos n\pi\,(1-\tfrac{1}{2}n^2\pi^2x^2+\tfrac{1}{3}n^2\pi^2x^3+\ldots).$$

Then, if we put $y_n = n\pi y$, where y is the series within the parentheses of (40), we see that

$$y_n{}^{-2}x^{-1}\sin y_n = \frac{(-1)^{n+1}}{n\pi}y^{-2}(1-\tfrac{1}{2}n^2\pi^2x^2+\tfrac{1}{3}n^2\pi^2x^3+\ldots).$$

Now, for brevity, let

$$X_1 = (1-x)^2 + x^2y_n^2,$$
$$X_2 = 1-x+x^2y_n^2,$$

and

$$X_3 = (1-\tfrac{1}{2}n^2\pi^2x^2+\tfrac{1}{3}n^2\pi^2x^3+\ldots).$$

Then, if we denote the n^{th} term of the series under the sign of summation in (38) by T_n, and differentiate $\log T_n$, the result is

$$\begin{aligned} \frac{1}{T_n}\frac{\partial T_n}{\partial x} &= -2y^{-1}\frac{dy}{dx} + X_1{}^{-1}\frac{dX_1}{dx} - X_2{}^{-1}\frac{dX_2}{dx} + X_3{}^{-1}\frac{dX_3}{dx} \\ &\quad - 2a^2t\lambda_n\frac{d\lambda_n}{dx} + r\cot\lambda_n r\,\frac{d\lambda_n}{dx}. \end{aligned}$$

But from (40), we have when $x=0$, $y=1$, and $\frac{dy}{dx}=-1$. Also, since $\lambda_n = \frac{y_n}{r_0}$,

$$\lambda_n = \frac{n\pi}{r_0} \quad\text{and}\quad \frac{d\lambda_n}{dx} = -\frac{n\pi}{r_0}.$$

Therefore, when $x=0$, we find

$$\frac{\partial T_n}{\partial x} = T_n\left[1+2t\left(\frac{an\pi}{r_0}\right)^2 - \frac{r}{r_0}n\pi\cot n\pi\frac{r}{r_0}\right]. \tag{42}$$

The succeeding complete values of the differential coefficients grow rapidly in complexity. Their derivation, however, is tedious rather than difficult. It will suffice, for our purposes, to write down the special value of the second differential coefficient, viz.:

$$\frac{\partial^2 T_n}{\partial x^2} = T_n \left\{ \begin{array}{l} 4 - n^2\pi^2 \left(1 + \frac{r^2}{r_0^2}\right) + 2t \left(\frac{an\pi}{r_0}\right)^2 \\ + 4t^2 \left(\frac{an\pi}{r_0}\right)^4 \\ - \left[2 \frac{r}{r_0} n\pi + 4a^2tr \left(\frac{n\pi}{r_0}\right)^3\right] \cot n\pi \frac{r}{r_0} \end{array} \right\}. \tag{43}$$

Now we observe that when $x = 0$,

$$T_n = \frac{(-1)^{n+1}}{n\pi} e^{-a^2t(n\pi/r_0)^2} \sin n\pi \frac{r}{r_0}, \tag{a}$$

$$T_n \frac{r}{r_0} n\pi \cot n\pi \frac{r}{r_0} = - r \frac{\partial T_n}{\partial r}{}^*, \tag{b}$$

$$2T_n t \left(\frac{an\pi}{r_0}\right)^2 = - 2t \frac{\partial T_n}{\partial t}, \tag{c}$$

$$T_n n^2\pi^2 \left(1 + \frac{r^2}{r_0^2}\right) = - (r_0^2 + r^2) \frac{\partial^2 T_n}{\partial r^2}, \tag{d}$$

$$4T_n t^2 \left(\frac{an\pi}{r_0}\right)^4 = + 4t^2 \frac{\partial^2 T_n}{\partial t^2}, \tag{e}$$

$$4T_n a^2rt \left(\frac{n\pi}{r_0}\right)^3 \cot n\pi \frac{r}{r_0} = + 4rt \frac{\partial^2 T_n}{\partial r \partial t}. \tag{f}$$

For brevity, put S for the value of the second member of (38) when $x = 0$, this value being the same as that given by (10), (19), or (20). Then, observing the effect of the sign of summation in (38), equations (42) and (43) in connection with the equivalents in (*a*) to (*f*) give

$$\frac{\partial (ru)}{\partial x} = S + r \frac{\partial S}{\partial r} - 2t \frac{\partial S}{\partial t}, \tag{44}$$

$$\frac{\partial^2 (ru)}{\partial x^2} = \left\{ \begin{array}{l} 4S - 2r \frac{\partial S}{\partial r} - 2t \frac{\partial S}{\partial t} \\ + 4rt \frac{\partial^2 S}{\partial r \partial t} + (r_0^2 + r^2) \frac{\partial^2 S}{\partial r^2} \\ + 4t^2 \frac{\partial^2 S}{\partial t^2} \end{array} \right\}. \tag{45}$$

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* To get the signs of the second members of (*b*) and (*f*), one must note that $\partial T_n / \partial r$ and $d\lambda_n / dx$ have opposite signs.

Finally, by Maclaurin's series we get

$$ru = S + \frac{\partial (ru)}{\partial x} x + \tfrac{1}{2} \frac{\partial^2 (ru)}{\partial x^2} x^2 + \dots \tag{46}$$

19. The partial differential coefficients involved in the second members of (44) and (45) are easily calculated. If we write, for brevity,

$$Q' = \sum_{n=1}^{n=\infty} \left\{ \begin{array}{l} e^{-(2m_0 n - m_0 - m)^2} \\ + e^{-(2m_0 n + m_0 + m)^2} \end{array} \right\}, \tag{47}$$

$$Q'' = \sum_{n=1}^{n=\infty} \left\{ \begin{array}{l} (2m_0 n - m_0 - m)\, e^{-(2m_0 n - m_0 - m)^2} \\ -(2m_0 n + m_0 + m)\, e^{-(2m_0 n + m_0 + m)^2} \end{array} \right\}, \tag{48}$$

$$Q''' = \sum_{n=1}^{n=\infty} \left\{ \begin{array}{l} (2m_0 n - m_0 - m)^2\, e^{-(2m_0 n - m_0 - m)^2} \\ +(2m_0 n + m_0 + m)^2\, e^{-(2m_0 n + m_0 + m)^2} \end{array} \right\}, \tag{49}$$

$$Q^{\text{iv}} = \sum_{n=1}^{n=\infty} \left\{ \begin{array}{l} (2m_0 n - m_0 - m)^3\, e^{-(2m_0 n - m_0 - m)^2} \\ -(2m_0 n + m_0 + m)^3\, e^{-(2m_0 n + m_0 + m)^2} \end{array} \right\}, \tag{50}$$

m_0 and m being defined by equations (18), we get from (23), (24), and (25)

$$r \frac{\partial S}{\partial r} = + r u_0 \left(1 - \frac{r_0}{a \sqrt{(\pi t)}} Q' \right), \tag{51}$$

$$2t \frac{\partial S}{\partial t} = - \frac{2 r_0 u_0}{\sqrt{\pi}} Q'', \tag{52}$$

$$(r_0^2 + r^2) \frac{\partial^2 S}{\partial r^2} = - \frac{r_0 (r_0^2 + r^2) u_0}{a^2 t \sqrt{\pi}} Q'', \tag{53}$$

$$4t^2 \frac{\partial^2 S}{\partial t^2} = + \frac{r_0 u_0}{\sqrt{\pi}} (6Q'' - 4Q^{\text{iv}}), \tag{54}$$

$$4rt \frac{\partial^2 S}{\partial r \partial t} = + \frac{r_0 r u_0}{a \sqrt{(\pi t)}} (2Q' - 4Q'''). \tag{55}$$

These will be applicable for all but the latest stages of cooling, and the corresponding derivatives from (10) will become applicable when these fail to be sufficiently converging.

The solution for conditioned cooling is thus made to depend on that for the ideal case of free cooling; and the expression (38), which is practically useless (on account of its slow convergence) for small values of the time is transformed into (46), which is available for any value of the time. We have limited the expansion to terms of the second order in x, believing that this is more than sufficient for any application of the theory to the earth; but the higher terms of the series may be easily derived if needed.

20. To complete the solution, it remains to show how to determine the temperature at the centre of the sphere. When we make $r = 0$, the expression for u from (46) assumes the ambiguous form $\frac{0}{0}$; but the usual process of differentiation gives

$$u = \frac{\partial S}{\partial r} + \frac{\partial \left[\frac{\partial (ru)}{\partial x} \right]}{\partial r} x + \tfrac{1}{2} \frac{\partial \left[\frac{\partial^2 (ru)}{\partial x^2} \right]}{\partial r} x^2 + \ldots \quad (56)$$
$$r = 0.$$

From equation (26) we have

$$\frac{\partial S}{\partial r} = u_0 \left[1 - \frac{4m_0}{\sqrt{\pi}} \sum_{n=1}^{n=\infty} \epsilon^{-m_0^2 (2n-1)^2} \right]. \quad (57)$$
$$r = 0.$$

The differential of (44) gives

$$\frac{\partial \left[\frac{\partial (ru)}{\partial x} \right]}{\partial r} = 2 \frac{\partial S}{\partial r} - 2t \frac{\partial^2 S}{\partial r \partial t}. \quad (58)$$
$$r = 0.$$

From (55) we have

$$2t \frac{\partial^2 S}{\partial r \partial t} = \frac{r_0 u_0}{a\sqrt{(\pi t)}} (Q' - 2Q'''), \quad (59)$$

in which, of course, r must be made zero for the special value in (58).

The special value of the differential of (45) with respect to r, required by the second term of (56), may be easily written down and worked out numerically for any assumed data, but the paucity of practical applications makes this work unnecessary.*

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*In note C of the supplement to his great work (see full title, p. 129) Poisson has summed the series (38) to terms of the first order in x. He does not separate his result into the term independent of and that dependent on x, but gives an expression for the sum of the two terms. He deduces several special results dependent on the assumption (a) that $x = 0$, or (b) that the strata whose temperatures are considered are near the surface of the sphere. Among these results is an expression for the temperature at the centre of the sphere under the assumption that $x = 0$. This expression agrees with our expression (26) or (57); but the casual reader would not observe the agreement; and, curiously enough, Poisson himself was unaware that the difference of two definite integrals in his expression is exactly zero. Using our notation,

21. Since no set of formulæ presenting any degree of complexity, or requiring many steps in their derivation, can be regarded as entirely trustworthy until they have been subjected to the test of actual application, we propose in the present and next sections to work out in detail a numerical example. For this purpose we shall assume such a value of the time that we can compute ru from (38) as well as from (46). The equivalence of the two forms will thus be tested, as well as the correctness of the signs of the several terms in the second members of (44) and (45).

We shall assume $m_0 = r_0 / 2a\sqrt{t} = \pi / 2$, a value used also in the illustrative example of §11. This gives

$$t = \left(\frac{r_0}{a\pi}\right)^2,$$

which, for the earth, if we use Sir W. Thomson's value of a^2, is, as stated in §3, about 100,000,000,000 years, With this value of the time, the first three terms of (38) will suffice to give ru in terms of $r_0 u_0$ to seven places of decimals. We shall also assume $x = 10^{-3}$ and $r = \frac{1}{2} r_0$ and compute first by means of (38).

22. To begin with, we need the first three roots of the transcendental equation (37). These we get from the first of (41), which gives with sufficient accuracy $\log y_n = \log(n\pi) + \log(1 - x) + \frac{1}{3}\mu (n\pi)^2 x^3$; μ being the modulus of the common system of logarithms. This gives

$$\log y_1 = 0.4967154,$$

$$\log y_2 = 0.797745,$$

$$\log y_3 = 0.973837.$$

For use in $\sin y_n$ and $\sin \lambda_n r$, it is best to compute the supplements $(n\pi - y_n)$.

he has for the fall of temperature at the centre of the sphere (see p. 52 of the supplement)

$$u_0 - u = \frac{2u_0}{\sqrt{\pi}} \left[\int_{m_0}^{\infty} e^{-z^2} dz + 2m_0 \sum_{n=1}^{n=\infty}{}' e^{-m_0^2 (2n-1)^2} - \sum_{n=1}^{n=\infty} \int_{-m_0}^{+m_0} e^{-(2m_0 n + z)^2} dz \right],$$

while our equation (26) gives only the middle term of this expression. But if in the last definite integral we replace $2m_0 n + z$ by z, the result is

$$\sum_{n=1}^{n=\infty}{}' \int_{m_0(2n-1)}^{m_0(2n+1)} e^{-z^2} dz = \int_{m_0}^{3m_0} e^{-z^2} dz + \int_{3m_0}^{5m_0} e^{-z^2} dz + \ldots$$

$$= \int_{m_0}^{\infty} e^{-z^2} dz,$$

which is identical with the first definite integral. Hence the above expression is the same as (26). Poisson, after reaching his equation, observes that

$$\int_{-m_0}^{+m_0} e^{-(2m_0 n + z)^2} dz < 2m_0 e^{-m_0^2 (2n-1)^2},$$

and proceeds to draw some conclusions which are qualitatively correct, but not susceptible of definite numerical evaluation.

These may be accurately computed from the second of (41), which gives in seconds of arc

$$\begin{aligned} \pi - y_1 &= 647''.97787, \\ 2\pi - y_2 &= 1295.983, \\ 3\pi - y_3 &= 1943.94. \end{aligned}$$

These and the above values of $\log y_n$ will be found to satisfy (37).

We can now compute all the factors in (38). The logarithms of the second algebraic factor and those of the factor $x^{-1} \sin y_n$ are found to be

$\log X_1 X_2{}^{-1}$	$\log x^{-1} \sin y_n$	
$9.9995655 - 10$,	0.4971479,	for $n = 1$,
$9.999565 - 10$,	0.798170,	$= 2$,
$9.999566 - 10$,	0.974252,	$= 3$.

Since $\lambda_n = y_n / r_0$, we have with our assumed value of t

$$e^{-a^2 t \lambda_n{}^2} = e^{-(y_n / \pi)^2},$$

and hence
$$\begin{aligned} \log e^{-a^2 t \lambda_n{}^2} &= 9.5665736 - 10, \text{ for } n = 1, \\ &= 8.266294 - 10, \qquad = 2, \\ &= 6.099162 - 10, \qquad = 3. \end{aligned}$$

Likewise, for the last factor we have $\sin \lambda_n r = \sin y_n r / r_0 = \sin \frac{1}{2} y_n$. Hence

$$\begin{aligned} \log \sin \tfrac{1}{2} y_n &= 9.9999995 - 10, \text{ for } n = 1, \\ &= 7.798170 - 10, \qquad = 2, \\ &= 9.999995_n - 10, \qquad = 3. \end{aligned}$$

Summing the logarithms of the factors of the several terms we get

	log	No.
1st term,	$9.0698557 - 10$,	$+ 0.11745073$,
2d	$4.965681 - 10$,	$+ .\ \ \ \ 0924$,
3d	$5.125300_n - 10$,	$- .\ \ \ \ 1334$.

The sum of the terms is 0.11744663. Multiplying this by 4 we get as the result from (38),

$$u = 0.4697865 u_0.$$

23. The details of the computation by equation (46) are as follows:—

The value of twice the first term, S, has already been computed in the example of §11; i. e.

$$S = 0.2341732 r_0 u_0,$$

using the larger of the two values there given, since a more extended computation shows that to be the more precise value.

We next need the values of the Q's which enter the second members of (51) to (55). Since by (18), $m = r / 2a\sqrt{t}$, and since for this example we have assumed $m_0 = r_0 / 2a\sqrt{t} = \pi / 2$ and $r = \frac{1}{2} r_0$, we have for use in (47) to (50)

$$\begin{array}{ll} m_0 - m = \frac{1}{4}\pi, & m_0 + m = \frac{3}{4}\pi, \\ 3m_0 - m = \frac{5}{4}\pi, & 3m_0 + m = \frac{7}{4}\pi, \\ 5m_0 - m = \frac{9}{4}\pi, & 5m_0 + m = \frac{11}{4}\pi, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array}$$

Hence we have, to terms of an order which may be neglected, the following system of equations in which the exponential quantities are factors respectively of the several coefficients in the vertical rows:

$$\begin{array}{llllll} Q' & = 1 \ e^{-(\frac{1}{4}\pi)^2} & + \ 1 \ e^{-(\frac{3}{4}\pi)^2} & + \ 1 \ e^{-(\frac{5}{4}\pi)^2} & + \ldots \\ Q'' & = \frac{1}{4}\pi & - \ \frac{3}{4}\pi & + \ \frac{5}{4}\pi & - \ldots \\ Q''' & = (\frac{1}{4}\pi)^2 & + (\frac{3}{4}\pi)^2 & + (\frac{5}{4}\pi)^2 & + \ldots \\ Q^{\text{iv}} & = (\frac{1}{4}\pi)^3 & - (\frac{3}{4}\pi)^3 & + (\frac{5}{4}\pi)^3 & - \ldots . \end{array}$$

These give, by an easy computation,

$$\begin{aligned} Q' &= + 0.54352, \\ Q'' &= + 0.41469, \\ Q''' &= + 0.35443, \\ Q^{\text{iv}} &= + 0.21069. \end{aligned}$$

Introducing these factors and the values of t and r in equations (51) to (55), we find the following equivalents, in which $r_0 u_0$ is a common factor of the second members: —*

$$\begin{aligned} r \frac{\partial S}{\partial r} &= + 0.01832 \, r_0 u_0, \\ 2t \frac{\partial S}{\partial t} &= - 0.46793, \\ (r_0^2 + r^2) \frac{\partial^2 S}{\partial r^2} &= - 2.886, \\ 4t^2 \frac{\partial^2 S}{\partial t^2} &= + 0.928, \\ 4rt \frac{\partial^2 S}{\partial r \partial t} &= - 0.293. \end{aligned}$$

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*The numerical values of all these differential coefficients have been checked by means of the equivalent forms obtained from equation (10).

By means of the last values and the above value of S we get from (44)

$$\frac{\partial\,(ru)}{\partial x} = +\,0.72042\, r_0 u_0.$$

Likewise from (45)

$$\frac{\partial^2\,(ru)}{\partial x^2} = -\,0.883\, r_0 u_0.$$

With these values and with $x = 10^{-3}$, we have from (46)

$$\begin{aligned} u &= 2u_0\,(0.2341732 + 0.00072042 - 0.00000044) \\ &= 0.4697864u_0. \end{aligned}$$

This agrees with the value obtained from (38) as closely as 7-place tables of logarithms will permit, the discrepancy being one unit in the last place.

It is scarcely necessary to remark in this connection that the precision of the preceding computations is of little importance except as a test of the correctness of the formulæ derived and as an illustration of the process which might be followed if our data for practical applications were sufficiently trustworthy. For all applications to the earth, computation of results by the preceding formulæ may be properly limited, in general, to three or four places of decimals, since defects in the best data at present attainable and probable inaccuracies in our fundamental assumptions (of §14), render those results uncertain even in the first place of decimals. For these reasons, also, and in view of the probability that the quantity x is less than 1 / 1000, it is believed that the solution for free cooling is an ample approximation in all applications to the earth.

24. In this and the following sections it is proposed to determine the amount of cubical contraction of a homogeneous sphere on the assumption of free cooling, and on the additional assumption that the coefficient of contraction is constant. These assumptions will give correctly the first approximation (or the term independent of x) in the case of a homogeneous sphere of moderately high initial temperature, and probably as good an approximation as is at present attainable for the secular contraction of the earth.*

If ϵ denote the coefficient of cubical contraction and ΔV the total change in volume during the time t after the initial epoch, we shall have

$$\Delta V = 4\pi\epsilon \int_0^{r_0} (u_0 - u)\, r^2 dr, \tag{60}$$

in which the symbols under the sign of integration have the same meaning as heretofore. The fall in temperature, $u_0 - u$, is given by equation (21) or (22), but for our present purposes it will be convenient to use another expression easily derived from (10). If in the latter, we make t first zero, and then t, and take the difference between the two expressions the result is

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*The consideration of the mechanical effects of secular contraction forms no part of our present work, though one of the principal objects of the analysis of this and the preceding paper is to furnish a basis for the investigation of those effects.

$$r(u_0 - u) = \frac{2r_0 u_0}{\pi} \sum_{n=1}^{n=\infty} \frac{(-1)^{n+1}}{n} \left(1 - e^{-a^2 t (n\pi / r_0)^2}\right) \sin n\pi \frac{r}{r_0}. \qquad (61)$$

Introducing this value of the fall in temperature of the shell considered, in (60) we get

$$\varDelta V = 8 r_0 u_0 \varepsilon \sum_{n=1}^{n=\infty} \frac{(-1)^{n+1}}{n} \left(1 - e^{-a^2 t (n\pi / r_0)^2}\right) \int_0^{r_0} r dr \sin n\pi \frac{r}{r_0}.$$

But the integral in this equation is equal to

$$-\frac{r_0^2}{\pi} \frac{\cos n\pi}{n} = \frac{r_0^2}{\pi} \frac{(-1)^{n+1}}{n}.$$

Therefore we have as the exact solution, under the assumed conditions,

$$\varDelta V = \frac{8 r_0^3 u_0 \varepsilon}{\pi} \sum_{n=1}^{n=\infty} \frac{1}{n^2} \left(1 - e^{-a^2 t (n\pi / r_0)^2}\right). \qquad (62)$$

When $t = 0$, this expression gives $\varDelta V = 0$, as it should. When $t = \infty$, the quantity under the sign of summation becomes

$$1 + \tfrac{1}{4} + \tfrac{1}{9} + \ . \ . \ . = \pi^2/6;$$

and hence $\varDelta V = \frac{4}{3}\pi r^3 u_0 \varepsilon,$

which is also obviously correct.

25. For large values of the time the series in (62) is evidently rapidly converging, but for small values the convergence is too slow for practical applications. It is desirable, therefore, to transform (62) and derive an expression from which we can compute the contraction when t is small, or during the earlier stages of cooling. To this end we proceed.

Referring again to the Gamma function of §4, and making the same substitutions for α and β as in §5, the series in (62) becomes

$$\tfrac{1}{6}\pi^2 - \frac{r_0}{a\sqrt{(\pi t)}} \int_0^\infty e^{-r_0^2 x^2 / 4a^2 t} dx \left(\cos \pi x + \tfrac{1}{4} \cos 2\pi x + \tfrac{1}{9} \cos 3\pi x + \ . \ . \ . \right),$$

in which x is merely a subject of integration and is, of course, not the x of

§§16 to 23. Now, for brevity, denote the trigonometric series in the last expression by σ_2, and let

$$\frac{d\sigma_2}{dx} = \sigma_1, \quad \frac{d^2\sigma_2}{dx^2} = \frac{d\sigma_1}{dx} = \sigma.$$

From the series in §6 we see that

$$\sigma = \tfrac{1}{2}\pi^2 - \frac{\frac{1}{2}(1 - g^2)\pi^2}{1 - 2g\cos \pi x + g^2} \text{ when } g = 1.$$

Multiplying by dx, integrating between the limits 0 and x, and making $g = 1$, we get

$$\sigma_1 = \tfrac{1}{2}\pi^2 (x - 1) \text{ for } 0 < x < 2.$$

Multiplying the last equation by dx and integrating again between 0 and x, there results

$$\sigma_2 = \pi^2 (\tfrac{1}{6} + \tfrac{1}{4}x^2 - \tfrac{1}{2}x).$$

This gives the value of σ_2 for all values of x from 0 to 2 both inclusive, or through the first period. For succeeding periods we have

$$\begin{aligned} \sigma_2 &= \pi^2 [\tfrac{1}{6} + \tfrac{1}{4}(x - 2)^2 - \tfrac{1}{2}(x - 2)], \text{ for } x \text{ between 2 and 4,} \\ &= \pi^2 [\tfrac{1}{6} + \tfrac{1}{4}(x - 4)^2 - \tfrac{1}{2}(x - 4)], \text{ for } x \text{ between 4 and 6,} \\ &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

Substituting these values of σ_2 in the above integral, and writing, for brevity, $m_0 = r_0 / 2a\sqrt{t}$, as in equation (18), we get

$$\tfrac{1}{6}\pi^2 - 2m_0\pi^{\frac{3}{2}} \left\{ \begin{aligned} &\int_0^2 e^{-m_0^2 x^2}\, dx\, (\tfrac{1}{6} + \tfrac{1}{4}x^2 - \tfrac{1}{2}x) \\ &+ \int_2^4 e^{-m_0^2 x^2}\, dx\, [\tfrac{1}{6} + \tfrac{1}{4}(x - 2)^2 - \tfrac{1}{2}(x - 2)] \\ &+ \int_4^6 e^{-m_0^2 x^2}\, dx\, [\tfrac{1}{6} + \tfrac{1}{4}(x - 4)^2 - \tfrac{1}{2}(x - 4)] \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \right\}.$$

Collecting and arranging the terms of this expression and performing the integrations so far as possible, it reduces to

$$\frac{\pi^2}{2m_0}\left(\frac{1}{\sqrt{\pi}} - \frac{1}{4m_0}\right) + \pi^{\frac{3}{2}} \sum_{n=1}^{n=\infty} \left[m_0^{-1} e^{-(2m_0 n)^2} - 2m_0 n (n + 1) \int_{2n}^{2n+2} e^{-m_0^2 x^2}\, dx \right].$$

Substituting this value for the quantity under the sign of summation in (62) and replacing x by z / m_0, we get finally

$$\Delta V = 8\pi r_0^3 u_0 \varepsilon \left\{ \begin{array}{c} \dfrac{1}{2m_0\sqrt{\pi}} - \dfrac{1}{8m_0^2} \\ + \sum\limits_{n=1}^{n=\infty} \left[\dfrac{1}{m_0\sqrt{\pi}} e^{-(2m_0 n)^2} - \dfrac{2}{\sqrt{\pi}} n(n+1) \displaystyle\int_{2m_0 n}^{m_0(2n+2)} e^{-z^2} dz \right] \end{array} \right\} . \quad (63)$$

$$m_0 = r_0 / 2a\sqrt{t}.$$

26. The series in the right-hand member of (63) converge with extreme suddenness for small values of the time. For the case of the earth they converge so rapidly that their first terms alone will suffice for the most accurate computations (one part in a million, say) when the time is as great as $r_0^2 / 4a^2$, which, with Sir W. Thomson's value of a^2, is about 273,000,000,000 years. Using this value of the time we have computed the amount of the contraction by both (62) and (63). The first three terms of (62) will suffice in this case for the precision attainable with 7-place logarithms. The second term in the first series of (63) or $e^{-16} / \sqrt{\pi}$ amounts to but six units in the eighth place of decimals; and the second term of the second series, or

$$\frac{12}{\sqrt{\pi}} \int_4^6 e^{-z^2} dz,$$

amounts to but twenty-eight units in the eighth place. Omitting other details of the computation, the results are

$$\Delta V = 0.1580729 . 8\pi r_0^3 u_0 \varepsilon \text{ from (62)},$$
$$= 0.1580728 . 8\pi r_0^3 u_0 \varepsilon \qquad (63).$$

These results serve to verify the numerical equivalence of (62) and (63), and show that they, like (46), or (10) and the group (19) to (22), bridge completely the time interval of cooling and enable the computer to give a definite answer to any question falling within the scope of the assumed conditions.

It is worthy of remark that the contraction just computed is about

$$\Delta V = 0.95 . \tfrac{4}{3}\pi r_0^3 u_0 \varepsilon ;$$

that is, in the time $t = r_0^2 / 4a^2$, a sphere will, under the assumed circumstances, attain ninety five per cent. of its possible contraction. The temperature at the centre of the sphere at the end of this time is (see §12) about one-sixth its initial value.

In all practical applications to the earth, for any but the latest stages of cooling, we may neglect the series in (63), which becomes then simply

$$\Delta V = 8\pi r_0^3 u_0 \varepsilon \left(\frac{a}{r_0} \sqrt{\frac{t}{\pi}} - \frac{a^2 t}{2 r_0^2} \right) . \qquad (64)$$

From this it appears that in the earlier stages of cooling the contraction is proportional to the square root of the time.